1 Supplemental material

In this section we will consider one gene at a time and so we will suppress the i subscript that was used in the main paper to index the genes. We also assume that a and b are known fixed parameters. We will use the notation $\mathfrak{L}$ to indicate liklihood density, and $\propto$ to indicate proportionality.

We begin by deriving the test statistic $\widetilde{\mathcal{F}}$ presented in equation (**??**).

Theorem 1 *Let ω be a subset of R^k, the corresponding residual sum of squares.. Then for the RVM model defined in section 2 of the paper, the liklihood ratio test statistic for testing $H_0 : \beta \in \omega$ against $H_1 : \beta \notin \omega$ will be of the form,*

$$\widetilde{\mathcal{F}} = \frac{n-k+2a}{r}\left(\frac{\widehat{\widehat{SS}} - \widehat{SS}}{\widetilde{SS}}\right), \tag{1}$$

where $\widehat{\widehat{SS}}$, $\widehat{SS}$ and $\widetilde{SS}$ are as defined in section **??**.

Proof. of Theorem 1

The liklihood function of an y under the linear model will be

$$\mathfrak{L}(y|\beta,\sigma) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left\|y - X\beta\right\|^2\right) \tag{2}$$

If let $z = \sigma^{-2}$ and include the distibutional assumptions we made on σ^{-2} this becomes

$$\mathfrak{L}(y|\beta,z) = \left(\frac{z}{2\pi}\right)^2 \exp\left(-\frac{z}{2}\left\|y - X\beta\right\|^2\right)\Gamma\left(z;a,b\right) \tag{3}$$

$$
\begin{aligned}
&= \left(\frac{z}{2\pi}\right)^2 \exp\left(-\frac{z}{2}\|y - X\beta\|^2\right) \frac{z^{a-1}\exp(-z/b)}{\Gamma(a)b^a} && (4)\\
&= c_1 z^{a+1} \exp\left(-z\left(\frac{1}{2}\|y - X\beta\|^2 + b^{-1}\right)\right) && (5)
\end{aligned}
$$

where c_1 is a scaling constant.

Under H_0 it can be shown that this quantity is maximized when

$$
\beta = \widehat{\widehat{\beta}} \text{ and } z = \frac{n + a - 1}{\widehat{\widehat{SS}} + 2b^{-1}} \tag{6}
$$

in which case the liklihood is

$$
\max_{z \in R^+, \beta \in \omega}\left(\mathfrak{L}(y|\beta, z)\right) = c_2\left(\widehat{\widehat{SS}} + 2b^{-1}\right)^{-(a+1)} \tag{7}
$$

When there is no restriction on β we can obtain a similar result for the maximized likelihood under H_1 in terms of the residual sum of squares

$$
\max_{z \in R^+, \beta \in R^k}\left(\mathfrak{L}(y|\beta, z)\right) = c_2\left(\widehat{SS} + 2b^{-1}\right)^{-(a+1)} \tag{8}
$$

Thus the liklihood ratio test will be to reject the H_0 when

$$
\left(\frac{\widehat{\widehat{SS}} + 2b^{-1}}{\widehat{SS} + 2b^{-1}}\right)^{a+1} > \delta \tag{9}
$$

where δ is chosen based on the size of the test. This is equivalent to

$$
\begin{aligned}
\frac{\widehat{\widehat{SS}} + 2b^{-1}}{\widehat{SS} + 2b^{-1}} &> \delta^{1/(a+1)} && (10)\\
\frac{\widehat{\widehat{SS}} + 2b^{-1} - \left(\widehat{SS} + 2b^{-1}\right)}{\widehat{SS} + 2b^{-1}} &> \delta^{1/(a+1)} - 1 && (11)\\
\frac{\widehat{\widehat{SS}} - \widehat{SS}}{\widetilde{SS}} &> \delta^{1/(a+1)} - 1 && (12)
\end{aligned}
$$

Thus we reject if

$$\widetilde{\mathcal{F}} = \frac{n-k+2a}{r}\left(\frac{\widehat{\widehat{SS}} - \widehat{SS}}{\widetilde{SS}}\right) > \frac{n-k+2a}{r}\left(\delta^{1/(a+1)} - 1\right) \tag{13}$$

Therefore .$\widetilde{\mathcal{F}}$ is a liklihood ratio test statistic ■

All that remains to complete the test is to find the distribution of the test statistic. In standard ANOVA theory we use the fact that $(n-k)\widetilde{\sigma}^2/\sigma^2$ is distributed according to a χ^2 with $n-k$ degrees of freedom, and that it is statistically independent of the value in the numerator of the ratio. Statistical independence of $\widetilde{\sigma}^2$ from the numerator follows immediately since $\widetilde{\sigma}^2$ depends on the data only through $\widehat{\sigma}$. In the theorem below we show that $(n-k+2a)\widetilde{\sigma}^2/\sigma$ is distributed according to a χ^2 with $n-k+2a$ degrees of freedom. Thus, the distribution of the revised test statistic is the same as before except for increased degrees of freedom. Addtionally, we we show that under the RVM model, the distribution of the the sample variances will be as in equation (**??**)

Theorem 2 *For $\widehat{\sigma}$ and $\widetilde{\sigma}$ as above,*

$$(n-k+2a)\frac{\widetilde{\sigma}^2}{\sigma^2} \sim \chi^2_{n-k+2a} \tag{14}$$

and

$$ab\left(\widehat{\sigma}_j^2\right) \sim F_{(n-k),2a} \tag{15}$$

Proof. From standard ANOVA theory we know that for fixed σ^2

$$\left((n-k)\frac{\widehat{\sigma}^2}{\sigma^2}\right) \sim \chi^2_{n-k} \tag{16}$$

Let $K = n-k$, then distribution of $\widehat{\sigma}^2$ is

$$\mathfrak{L}\left(\widehat{\sigma}^2\right) = \frac{K}{\sigma^2\Gamma(K/2)2^{K/2}}\left(\frac{K\widehat{\sigma}^2}{\sigma^2}\right)^{K/2-1}\exp\left(\frac{-K\widehat{\sigma}^2}{2\sigma^2}\right) \tag{17}$$

if we let $z = \sigma^{-2}$, then the joint distribution of σ^2 and z is

$$\begin{aligned}
\mathfrak{L}\left(\widehat{\sigma}^2, z\right) &= \frac{zK}{\Gamma(K/2)2^{K/2}}\left(zK\widehat{\sigma}^2\right)^{K/2-1}\exp\left(-\frac{zK\widehat{\sigma}^2}{2}\right)\left(\frac{z^{a-1}\exp(-z/b)}{\Gamma(a)b^a}\right) && (18)\\
&\propto \left(\widehat{\sigma}^2\right)^{K/2-1}\exp\left(-z\left(\frac{K}{2}\widehat{\sigma}^2+b^{-1}\right)\right)\left(z^{K/2+a-1}\right) && (19)\\
&\propto \left(\widehat{\sigma}^2\right)^{K/2-1}\left(\frac{K}{2}\widehat{\sigma}^2+b^{-1}\right)^{1-(K/2+a)} && (20)\\
&\quad \cdot\exp\left(-z\left(\frac{K}{2}\widehat{\sigma}^2+b^{-1}\right)\right)\left(z\left(\frac{K}{2}\widehat{\sigma}^2+b^{-1}\right)^{K/2+a-1}\right) && (21)
\end{aligned}$$

If we let

$$u = z\left(K\widehat{\sigma}^2+2b^{-1}\right) = (n-k+2a)\frac{\widetilde{\sigma}^2}{\sigma^2} \tag{22}$$

and

$$v = ab\widehat{\sigma}^2 \tag{23}$$

then after a change of varaibles

$$\begin{aligned}
\mathfrak{L}(u,v) &\propto (v)^{K/2-1}\left(\frac{K}{a}v+1\right)^{-(K/2+a)} && (24)\\
&\quad \cdot\exp\left(-\frac{u}{2}\right)\left(\frac{u}{2}\right)^{(K+2a)/2-1} && (25)\\
&\propto F_{2a,K}(v)\cdot\chi^2_{K+2a}(u) && (26)
\end{aligned}$$

so if we integrate over all possible v we find that

$$\mathfrak{L}\left(u\right) = \chi^2_{K+2a}\left(u\right) \tag{27}$$

and integrating over all possible u gives

$$\mathfrak{L}\left(v\right) = F_{2a,K}(v) \tag{28}$$

■